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# Electrodynamics of a mechano-driven media system moving with acceleration

Zhong Lin Wang*

1. Beijing Institute of Nanoenergy and Nanosystems, Chinese Academy of Sciences, Beijing 101400, P. R. China
2. School of Nanoscience and Technology, University of Chinese Academy of Sciences, Beijing 100049, P. R. China
3. School of Materials Science and Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0245, USA.

* Corresponding author: zhong.wang@mse.gatech.edu

**Abstract**

In classical electrodynamics, by motion for either the observer or the media, it always means that the relative moving velocity is a constant along a straight line. In such an inertia system, it is relatively easy to describe the electromagnetic behavior using special relativity. However, for engineering applications, the media may move with acceleration in non-inertial frame. In this case, we have developed an approximated *Maxwell's equations for a mechano-driven slow-moving media system* by neglecting relativistic effect. We also elaborate the conditions under which the Maxwell's equations for a mechano-driven system converges to the standard Maxwell's equations and the Lorentz transformation is equivalent to the Galilean transformation. We found that the standard differential forms of Maxwell equations hold in inertia frame, but in non-inertia frame, the differential forms of the Faraday's law of electromagnetic induction and the Ampere-Maxwell law may be subjected to be expanded in several cases. Lastly, the theory is presented in Lagragian scheme, from which the Maxwell's equations for a mechano-driven media is derived.

Keywords: Maxwell's equations for mechano-driven slow-moving media, Galilean transformation, Faraday's law of electromagnetic induction, non-inertia reference frame

## 1. Introduction

In classical electrodynamics, describing the electromagnetic phenomenon occurring in a Lab reference frame by an observer who is moving has been naturally assumed that the moving velocity is a constant, which is the case for special relativity. This is the taken as is tradition or habit of many researchers even without specifically defining the moving velocity [1, 2]. The fields in the two reference frames are correlated by the Lorentz transformation to warrant the covariance of the Maxwell's equations. In general, physics phenomena should be the same when viewed by two observers moving at a constant relative velocity, e.g., the inertial frames are equivalent with each other because of the exclusion of external forces. In classical text books for electrodynamics, special relativity is the only available text for students to learn about moving media.

However, in practice, most of the phenomena are being observed in non-inertia frame with the observer or media moving with an acceleration, such as circular motion, repeated oscillation etc. Once the movement is in a non-inertia frame, there must be the contribution from an external force that drives the media to move. If the moving media has surface electrostatic charges due to effect such as triboelectrification [3], the non-uniform motion of the media must include the input work made by the external force. Although the total energy is conservative, the total electromagnetic energy is not in such a non-inertia frame, so that the Maxwell's equations may not be covariant. To develop an effective approach that can describe the electrodynamics of a media system that moves at a varying velocity, we have recently developed the *Maxwell's equations for mechano-driven slowing-moving media* [4, 5], in which the media have a time-dependent shapes, volumes and boundaries, and more importantly, the movement of the media is described by a velocity field that is time and space dependent, $\boldsymbol{v}(\boldsymbol{r}, t)$. The only required condition is that the moving speed is much less than the speed of light ($v$ << c) by using the Galilean transformation. The equations are to describe the coupling and interactions among (mechano) force - electric – magnetic fields.

In this paper, we tend to answer three questions: conditions under which the Lorentz transformation can be approximated as Galilean transformation; the Maxwell's equations for mechano-driven system with the inclusion of constitutive relations; and the conditions under which the Maxwell's equations for a mechano-driven system may pertain the format of the standard Maxwell's equations. We then studied the extension of the Faraday's law of electromagnetic induction for media that are moving with an acceleration.

## 2. Electrodynamics of a moving media system in inertia reference frame

In physics, a reference frame is always chosen for describing the physics laws in mathematical forms, and the inertia frame is the simplest one. The mathematical expression of a physics law should take the same format in all inertia reference frames. In electrodynamics, one reference frame is in the Lab frame where the observer is at rest. The other reference frame is attached to the medium that is moving, which an inertia frame if the medium movement has no acceleration and it is a non-inertial frame if the medium moving moves with an acceleration.

Someone may say that the field due to a moving charge can be calculated precisely using the Li énard-Wiechert potential even it has an acceleration, which is correct because the charge is a point charge without volume and boundary. The moving charge can be viewed as a pulsed current density in space that is represented using a delta function. For a moving medium that has a surface and volume with certain permittivity, the field due to a moving medium cannot be calculated using the Liénard-Wiechert potential. The solution has to meet the boundary conditions on the medium surface. This different must be kept in mind in order to avoid any further misunderstanding. All of our discussions are about medium that has a volume and boundary. A medium is an assembly of atoms in specific order and chemistry so that it has certain dielectric, electric and elastic properties, so that it can have different electrical, optical, thermal and mechanical properties. A medium is NOT just an aggregation of charges!

### 2.1 Conditions under which the Lorentz transformation approaches the Galilean transformation

In classical electrodynamics and field theory, once it says "motion media", it usually means a medium that moves at a constant velocity along a straight line. This has been naturally taken as given in almost all text books. In such an inertia system that do not have other form of energy exchange with the outside system, special relativity is most powerful for treating the electromagnetic behavior observed in the Lab frame (at rest) $(\boldsymbol{r}, t)$ $(\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{D}, \boldsymbol{H})$ and the moving frame affixed to the medium $(\boldsymbol{r}', t')$ $(\boldsymbol{E}', \boldsymbol{B}', \boldsymbol{D}', \boldsymbol{H}')$ that is moving at a constant velocity $v_0$ along the x-axis, which are correlated by the Lorentz transformation (Fig. 1a):

$$x' = \gamma(x - v_0 t) \tag{1a}$$

$$y' = y \tag{1b}$$

$$z' = z \tag{1c}$$

$$t' = \gamma(t - x v_0/c^2) \tag{1d}$$

where $\gamma = 1/(1 - v_0^2/c^2)^{1/2}$, and $c$ is the speed of light in vacuum. In the relativistic theory, space and time are unified and correlated. Under the condition of $v_0 \ll c$, the fields in the two reference frames are related by (Fig. 1a):

$$\boldsymbol{E}' \approx \boldsymbol{E} + \boldsymbol{v}_0 \times \boldsymbol{B} \tag{2a}$$

$$\boldsymbol{B}' \approx \boldsymbol{B} - \boldsymbol{v}_0 \times \boldsymbol{E}/c^2 \tag{2b}$$

$$\boldsymbol{D}' \approx \boldsymbol{D} + \boldsymbol{v}_0 \times \boldsymbol{H}/c^2 \tag{2c}$$

$$\boldsymbol{H}' \approx \boldsymbol{H} - \boldsymbol{v}_0 \times \boldsymbol{D} \tag{2d}$$

$$\boldsymbol{J}_f' \approx \boldsymbol{J}_f - \rho_f \boldsymbol{v}_0 \tag{2e}$$

$$\rho_f' \approx \rho_f - \boldsymbol{v}_0 \cdot \boldsymbol{J}/c^2 \tag{2f}$$

In the Galilean transformation:

$$x' = x - v_0 t \tag{3a}$$

$$y' = y \tag{3b}$$

$$z' = z \tag{3c}$$

$$t' = t \tag{3d}$$

where the space and time are absolutely independent. Galilean space and time are most frequently used in classical physics and engineering applications. The fields in the two reference frames are related by:

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}_0 \times \boldsymbol{B} \tag{4a}$$

$$\boldsymbol{B}' = \boldsymbol{B} \tag{4b}$$

$$\boldsymbol{D}' = \boldsymbol{D} \tag{4c}$$

$$\boldsymbol{H}' = \boldsymbol{H} - \boldsymbol{v}_0 \times \boldsymbol{D} \tag{4d}$$

$$\boldsymbol{J}_f' \approx \boldsymbol{J}_f - \rho_f \boldsymbol{v}_0 \tag{4e}$$

$$\rho_f' \approx \rho_f \tag{4f}$$

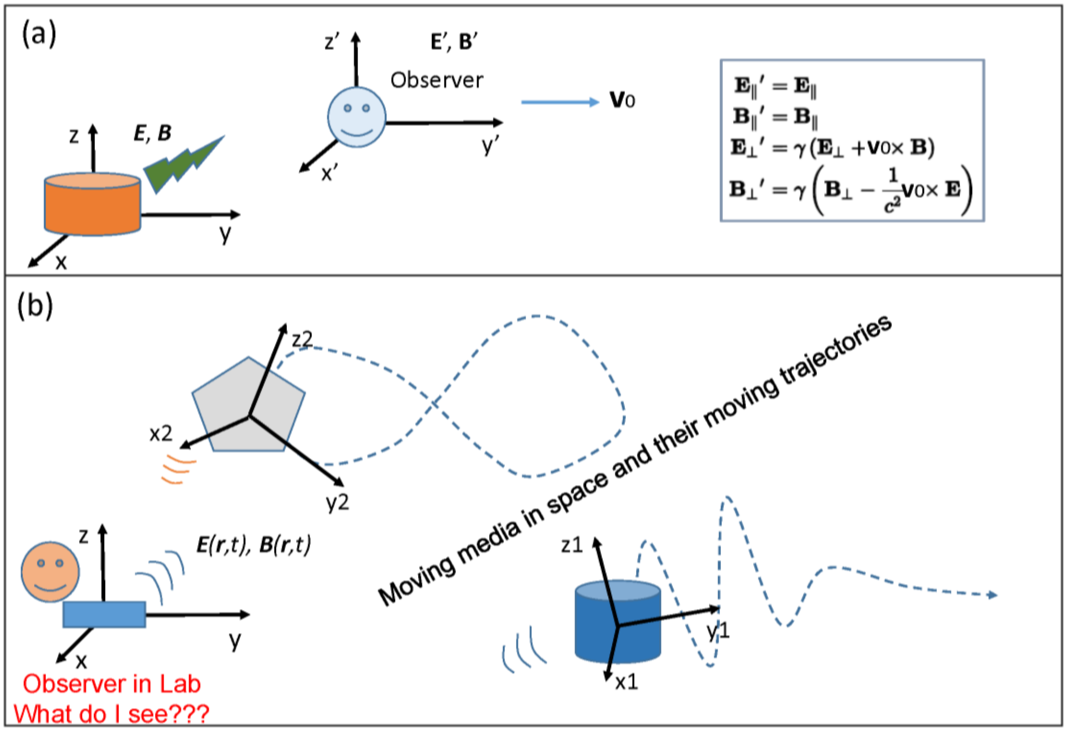


Figure 1. (a) Case for special relativity: Schematic diagram representing the general approach in special relativity, in which the fields observed by an observer who is moving at a constant velocity v about the fields ($\mathbf{E}'$, $\mathbf{B}'$) that was first generated in the Lab frame ($\mathbf{E}$, $\mathbf{B}$). The Lorentz transformed relationships between the fields in the two reference frame in parallel to $\boldsymbol{v}$ and perpendicular to $\boldsymbol{v}$ are inset. (b) Mechano-driven moving media system in non-inertia frame: A general case in which the observer is on the ground frame (called Lab frame), with several media moving at complex velocities along various trajectories as represented by the dashed lines. The medium can translate, rotate, expand and even split. The co-moving frames for the media are: (x1, y1, z1), (x2, y2, z2). In such a case, the Lorentz transformation for special relativity cannot be easily applied, and the only realistic approach is to *express all of the fields in the frame where the observation is done (Lab frame) and all of the fields are expressed in the variables in the same frame*.

The conditions under which the Lorentz transformation is equivalent to Galilean transformation are as following [6]:

1. The relative speed between two inertial frames of reference is much smaller than the speed of light in vacuum: $v_0 \ll c$; and

2. Galilean phenomenon takes place in an arena, the spatial extension of which is much smaller than the distance traveled by light during the duration of the phenomenon: $x \ll ct$.

With considering that 9 times of speed of sound is ~3 km/s, which is 1/100,000 of the speed of light, it can be safely stated that for the macroscopic object on earth, the above two conditions are absolutely satisfied, so that Galilean transformation is valid for practical applications. However, in universe, light takes 180 s to travel from earth to Mars, and it takes 100,000 light years to travel across the milky way galaxy. Therefore, the movement of light in universe can be viewed as the moving of a turtle on earth, so slow in comparison to the vast space. Therefore, Lorentz transformation is needed for outer space.

### 2.2 Constitutive relations

Due to the movement of the media, the constituent relations derived under low-speed Lorentz transformation has to be used [7]:

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} - \varepsilon \boldsymbol{v}_0 \times \boldsymbol{B} \quad (5a)$$

$$\boldsymbol{H} = \boldsymbol{B}/\mu + \varepsilon \boldsymbol{v}_0 \times \boldsymbol{E} \quad (5b)$$

It must be pointed out that the constitute relations were derived by assuming the covariance of the Maxwell equations in inertia frame. For inertia frame, the covariance of the Maxwell's equations may not be preserved due to the presence of other forces; in such a case, the constitutive relations may not hold.

## 3. Electrodynamics of a moving media system in non-inertia reference frame – an approximated approach

Most of the existing work assumes that the moving velocity of the media is a constant and along a straight line, especially for special relativity. As for practical applications, physical phenomena are the same when viewed by two observers moving with a constant velocity $\boldsymbol{v_0}$ relative to one another, provided the coordinate in the space and time for the two frames are related by the Galilean transformation: $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}_0 t$, and $t' = t$. But the situation changes if the motion frame is non-inertia

(Fig. 1b). For a medium that moves along a complex trajectory, there is no simple approach for using Lorentz transformation!

In order to consider the case for motion with acceleration in a general case, we need to start from the integral forms of the four physics laws, and assume that the media moves at an arbitrary low velocity $\boldsymbol{v}(\boldsymbol{r}, t)$. We now start from the integral form of the Faraday's electromagnetic induction law as stated in [1]:

$$\oint_C \boldsymbol{E}' \cdot d\boldsymbol{L} = -\frac{d}{dt}\iint_C \boldsymbol{B} \cdot d\boldsymbol{s} \tag{6}$$

where $\boldsymbol{E'}$ is the electric field on the moving medium in the frame where $d\boldsymbol{L}$ is at rest. The starting point of this equation is different from that we used in [4, 5]. What we have taken in [4,5] is mainly toward engineering applications. The approach taken in Eq. (6) is from the point of view of field theory. The physics understanding about electromagnetic law is that the decreasing rate of the total magnetic flux through an open surface $\boldsymbol{s}$, is the work done by the electromotive force on a point charge around the edge-loop of the surface $\boldsymbol{s}$. The right hand-side of Eq. (6) can be accurately calculated using the flux theorem as follows

$$\frac{d}{dt}\iint_{\boldsymbol{s}(t)} \boldsymbol{B} \cdot d\boldsymbol{s} = \iint_{\boldsymbol{s}(t)}\{\frac{\partial}{\partial t}\boldsymbol{B} + [\nabla \cdot \boldsymbol{B}]\boldsymbol{v} - \nabla\times[\boldsymbol{v}\times\boldsymbol{B}]\} \cdot d\boldsymbol{s} = \iint_{\boldsymbol{s}(t)}\{\frac{\partial}{\partial t}\boldsymbol{B} - \nabla\times[\boldsymbol{v}\times\boldsymbol{B}]\} \cdot d\boldsymbol{s} \tag{7}$$

Note, here $\boldsymbol{r}$ and $t$ are treated as independent variables, rather than assuming $\boldsymbol{r}(t)$. Substitution of Eq. (7) into Eq. (6) and use the Stokes theorem, we have

$$\nabla\times (\boldsymbol{E}' - \boldsymbol{v}\times \mathrm{B}) = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{8}$$

This is valid for a general velocity $\boldsymbol{v}$. Alternatively, the Lorentz force acting on a point charge $q$ in Lab frame is $\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$, and the force acting on the same charge in its rest frame is $\boldsymbol{F}' = q\boldsymbol{E}'$.

In general, $\boldsymbol{F} \neq \boldsymbol{F}'$ because of the accelerated movement of the reference frame, unless the moving frame is *an inertia frame* [8], which means that $\boldsymbol{v} = \boldsymbol{v}_0$. Therefore, only in inertia frame that the movement is at a constant speed along a straight line, $\boldsymbol{v} = \boldsymbol{v}_0$, we can have $\boldsymbol{F}' = q\boldsymbol{E}'$, which gives $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}_0\times\boldsymbol{B}$ in the inertial frame, we have

$$\nabla\times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{9}$$

Therefore, we must emphasize that the standard mathematical expression of the *Faraday electromagnetic induction law in differential form as stated in Eq. (9) holds for moving media only if* *its moving velocity is a constant and along a straight line*! e.g., in inertia frame! Therefore the conclusion reached by Sheng et al. is valid only if the moving velocity $\boldsymbol{v}_0$ is constant [9]. For a general case of non-inertia frame, the differential form Eq. (8) is more general. By the same token, we also believe that the mathematical expressions for the differential form of Ampere-Maxwell law may holds only for inertial frame. This concept can be generalized as: the differential form of the Maxwell's equations may hold only in inertia frame, and it is subject to be expanded for non-inertia frame/motion [4, 5].

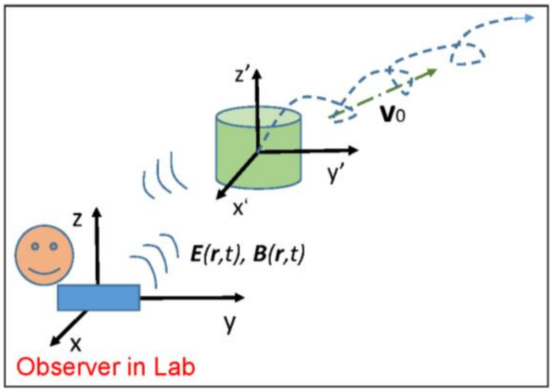


Figure 2. Schematic diagram showing an approximated method for decompose a non-inertia movement as a movement in an inertia frame plus a correction in non-inertia frame.

If the moving velocity of the medium is a constant for inertial frame , the relationship can be easily derived from special relativity under slow speed limit as $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}_0 \times \boldsymbol{B}$. But for non-inertial frame, the media moves at an arbitrary low velocity with acceleration, we can adopt an approximate approach, in which the motion velocity is split into a constant component $\boldsymbol{v}_0$ and a time-dependent component $\delta\boldsymbol{v}$,

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}_0 + \delta\boldsymbol{v}(\boldsymbol{r},t) \tag{10}$$

where the term $\delta\boldsymbol{v}(\boldsymbol{r},t)$contains both rotation and small component of rigid translation. If the local field can be approximately treated as $\boldsymbol{E}' \approx \boldsymbol{E} + \boldsymbol{v}_0 \times \boldsymbol{B}$ by only considering the constant velocity component, which is schematically illustrated in Fig. 2. The idea is that we try to use the results developed for an inertial reference frame for approximately treating the case in a non-inertia reference frame,

$$\oint_C \boldsymbol{E}' \cdot d\boldsymbol{L} \approx \oint_C (\boldsymbol{E} + \boldsymbol{v}_0 \times \boldsymbol{B}) \cdot d\boldsymbol{L} = \iint_{\boldsymbol{s}(t)} \nabla\times [\boldsymbol{E} + \boldsymbol{v}_0 \times\boldsymbol{B}] \cdot d\boldsymbol{s} \tag{11}$$

This equation means that the changing rate of the flux of the magnetic field through an open surface plus the work done by the Lorentz force due to media movement on a unit charge around its edge loop is the circulation of the induced electromotive force around its closed edge loop. Therefore:

$$\nabla\times (\boldsymbol{E} - \delta\boldsymbol{v}\times\mathrm{B}) \approx -\frac{\partial}{\partial t}\boldsymbol{B} \tag{12}$$

By the same token, in analogy to the Faraday's electromagnetic induction law, we can start from the integral form of the Ampere-Maxwell law in the same fashion:

$$\oint_C \boldsymbol{H}' \cdot d\boldsymbol{L} = \iint_C \boldsymbol{J}_f \cdot d\mathbf{s} + \frac{d}{dt}\iint_C \boldsymbol{D} \cdot d\mathbf{s} \tag{13}$$

where the $\boldsymbol{H'}$ is the magnetic field on the moving medium in the frame where $d\boldsymbol{L}$ is at rest. Using the flux theorem, we have

$$\oint_C \boldsymbol{H}' \cdot d\boldsymbol{L} = \iint_C (\boldsymbol{J}_f + \rho_f \boldsymbol{v}) \cdot d\mathbf{s} + \iint_C \frac{\partial}{\partial t}\boldsymbol{D} \cdot d\mathbf{s} - \oint_C (\boldsymbol{v}\times\boldsymbol{D}) \cdot d\boldsymbol{L} \tag{14}$$

This equation means that the sum of, the total current flowing through an open surface and the current produced by the free charges due to media movement (right-hand side first term), the changing rate of the electric flux through the open surface (right-hand side second term), and the electric circulation produced by media movement (right-hand side third term), is the circulation of the magnetic field around its closed edge loop. Using the Stokes's theorem, Eq. (14) becomes

$$\nabla\times (\boldsymbol{H}' + \boldsymbol{v}\times\boldsymbol{D}) = \boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{D} \tag{15}$$

Note, $\boldsymbol{H'}$ is the magnetic field in the moving frame affixed to the media that moves at an arbitrary low velocity $\boldsymbol{v}$.

Under low speed limit and in inertia frame, the local magnetic field in the moving frame is $\boldsymbol{H'} \approx \boldsymbol{H} - \boldsymbol{v}_0 \times \boldsymbol{D}$ (Eq. (2d), we have

$$\nabla\times (\boldsymbol{H} + \delta\boldsymbol{v}\times\boldsymbol{D}) \approx \boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{D} \tag{16}$$

The terms of $\delta\boldsymbol{v}\times\boldsymbol{B}$ and $\delta\boldsymbol{v}\times\boldsymbol{D}'$ are the sources of produced electromagnetic waves by media movement. In the case of the moving velocity is constant, $\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}_0$, $\delta\boldsymbol{v} = 0$, Eqs. (12, 16) resume that of the Standard Maxwell equations. This is the expected result from the Lorentz covariance for inertial frame.

By introducing the approximate constitutive relations for non-inertia frame:

$$\boldsymbol{D} \approx \varepsilon\boldsymbol{E} - \varepsilon\boldsymbol{v}_0\times\boldsymbol{B} \tag{17a}$$

$$\boldsymbol{H} \approx \boldsymbol{B}/\mu + \varepsilon\boldsymbol{v}_0\times\boldsymbol{E} \tag{17b}$$

The Maxwell equations for a general case can be stated as follows:

$$\varepsilon\nabla\cdot\boldsymbol{E} = \rho_f + \varepsilon\nabla\cdot(\boldsymbol{v}_0\times\boldsymbol{B}) \tag{18a}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{18b}$$

$$\nabla\times(\boldsymbol{E} - \delta\boldsymbol{v}\times\boldsymbol{B}) = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{18c}$$

$$\nabla\times(\boldsymbol{B}/\mu + \varepsilon\boldsymbol{v}_0\times\boldsymbol{E} + \varepsilon\delta\boldsymbol{v}\times\boldsymbol{E}) = \boldsymbol{J}_f + \rho_f\boldsymbol{v} + \frac{\partial}{\partial t}(\varepsilon\boldsymbol{E} - \varepsilon\boldsymbol{v}_0\times\boldsymbol{B})$$

$$\approx \boldsymbol{J}_f + \rho_f\boldsymbol{v} + \varepsilon\frac{\partial}{\partial t}\boldsymbol{E} + \varepsilon\boldsymbol{v}_0\times(\nabla\times\boldsymbol{E})$$

Since

$$\nabla\times(\boldsymbol{v}_0\times\boldsymbol{E}) = \boldsymbol{v}_0(\nabla\cdot\boldsymbol{E}) - (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{E}$$

$$\nabla(\boldsymbol{v}_0\cdot\boldsymbol{E}) = (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{E} + \boldsymbol{v}_0\times(\nabla\times\boldsymbol{E})$$

$$\boldsymbol{v}_0\times(\nabla\times\boldsymbol{E}) = \nabla(\boldsymbol{v}_0\cdot\boldsymbol{E}) - (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{E} = \nabla(\boldsymbol{v}_0\cdot\boldsymbol{E}) + \nabla\times(\boldsymbol{v}_0\times\boldsymbol{E}) - \rho_f\boldsymbol{v}_0/\varepsilon$$

We have

$$\nabla\times(\boldsymbol{B}/\mu + \varepsilon\delta\boldsymbol{v}\times\boldsymbol{E}) = \boldsymbol{J}_f + \rho_f\delta\boldsymbol{v} + \varepsilon\nabla(\boldsymbol{v}_0\cdot\boldsymbol{E}) + \varepsilon\frac{\partial}{\partial t}\boldsymbol{E} \tag{18d}$$

It must be pointed out that the quantities in Eqs. (18a-d) derived directly from the integral Maxwell's equations are given in Lab frame ($\boldsymbol{r}$, $t$).

## 4. Mechano-polarization introduced by media relative movement

For moving media, it is possible to have physical contact among the moving media, so that their surfaces must have electrostatic charges due to contact electrification (triboelectric effect) and/or piezoelectric effect. Thus, a variation in medium shape and/or moving medium object results in not only a local time-dependent charge density $\rho_s$, but also a local “virtual” electric current density due to the media movement. To account for both terms, we have introduced a *mechano-induced polarization* vector $\boldsymbol{P}_s$, so that the displacement vector $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ is replaced by

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} + \boldsymbol{P}_s \tag{19}$$

Therefore, all of the above equations are held except replacing $\varepsilon\boldsymbol{E}$ by $\varepsilon\boldsymbol{E} + \boldsymbol{P}_s$ [10, 11].

$$\nabla \cdot (\varepsilon\boldsymbol{E} + \boldsymbol{P}_s) = \rho_f + \varepsilon\nabla \cdot (\boldsymbol{v}_0 \times \boldsymbol{B}) \tag{20a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{20b}$$

$$\nabla \times (\boldsymbol{E} - \delta\boldsymbol{v} \times \boldsymbol{B}) = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{20c}$$

$$\nabla \times [\boldsymbol{B}/\mu + \delta\boldsymbol{v} \times (\varepsilon\boldsymbol{E} + \boldsymbol{P}_s)] = \boldsymbol{J}_f + \rho_f\,\delta\boldsymbol{v} + \varepsilon\,\nabla[\boldsymbol{v}_0 \cdot (\varepsilon\boldsymbol{E} + \boldsymbol{P}_s)] + \frac{\partial}{\partial t}(\varepsilon\boldsymbol{E} + \boldsymbol{P}_s) \tag{20d}$$

## 5. Faraday’s law of electromagnetic induction for a medium moving with acceleration

### 5.1 Extension of Faraday’s law of electromagnetic induction in a moving large conductive media without considering inertia force

As for Faraday’s law, let’s use a conductive loop to replace the medium we have been referring above for easy discussion. The Faraday’s law as stated in Eq. (6) can be physically understood as the work done on a unit charge by the electromotive force is the reducing rate of the magnetic flux through a closed loop circuit. Now let’s consider the movement of the unit charge in the circuit, the moving velocity of which is composed of two components:

$$\boldsymbol{v}_t = \boldsymbol{v} + \boldsymbol{v}_r, \tag{21}$$

where $\boldsymbol{v}$ is the moving velocity of the loop circuit, e.g. the moving velocity of the medium, $\boldsymbol{v}_r$ is the relative moving velocity of the unit charge in reference to the circuit. This term is important if the conductive medium is large so that we have to consider the additional contribution of the charge moving velocity with respect to the medium. Now if we only consider the electric and Lorentz force acting on the unit charge moving along the circuit, we have

$$\boldsymbol{E}' \approx \boldsymbol{E} + \boldsymbol{v}_t \times \boldsymbol{B} \tag{22}$$

Using Eq. (7), Eq. (6) becomes

$$\oint_C [\boldsymbol{E} + \boldsymbol{v}_r \times \boldsymbol{B}] \cdot d\boldsymbol{L} \approx -\iint_C \frac{\partial}{\partial t}\boldsymbol{B} \cdot d\boldsymbol{s} \tag{23}$$

There are two possible cases from Eq. (23):

a) The integral path is a thin wire without intercepting with any conductive medium (Fig. 3a), so that the relative moving velocity term in Eq. (23) is parallel to the integral path, so that the term [ $\boldsymbol{v}_r \times \boldsymbol{B}] \cdot d\boldsymbol{L}$ drops out in the integral, the equation resumes the original form of the Faraday's law:

$$\nabla \times \boldsymbol{E} \approx -\frac{\partial}{\partial t}\boldsymbol{B} \tag{24}$$

This is the case presented in all of the text books, but it has a condition of thin wire approximation.

b) If the conductive medium is large so that the relative moving velocity of the unit charge has a component perpendicular to the integral path in the segment where the integral loop intercepts with the conduction medium, such as a loop circuit, part of which is made of a rotating metal disc as shown in Fig. 3b [12]. In such a case, the charge can wonder in the conductive disc, and the relative moving velocity may not be parallel to the integral path, thus, we may have $\oint_C [\boldsymbol{v}_r \times \boldsymbol{B}] \cdot d\boldsymbol{L} \neq 0$. Using the Stoker's law, for the space inside the conductive medium, Eq. (23) becomes

$$\nabla \times (\boldsymbol{E} + \boldsymbol{v}_r \times \boldsymbol{B}) \approx -\frac{\partial}{\partial t}\boldsymbol{B} \tag{25}$$

Outside of the media, the governing equation is Eq. (24). This case is rarely discussed in classical text books except a recent one [13]. Eq. (25) can be applied to cases that have a fan-blade shape rotating media and a rotation metal disc in a magnetic field. The conditions that the two cases converge is $\oint_C [\boldsymbol{v}_r \times \boldsymbol{B}] \cdot d\boldsymbol{L} = 0$ [14]. The Ampere-Maxwell's law could be given in an analogous way:

$$\nabla \times (\boldsymbol{H} - \boldsymbol{v}_r \times \boldsymbol{D}) = \boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{D} \tag{26}$$

Both Eqs. (25) and (26) apply to the space inside the medium. Outside the medium in vacuum, the equations resume the original Maxwell's equations.

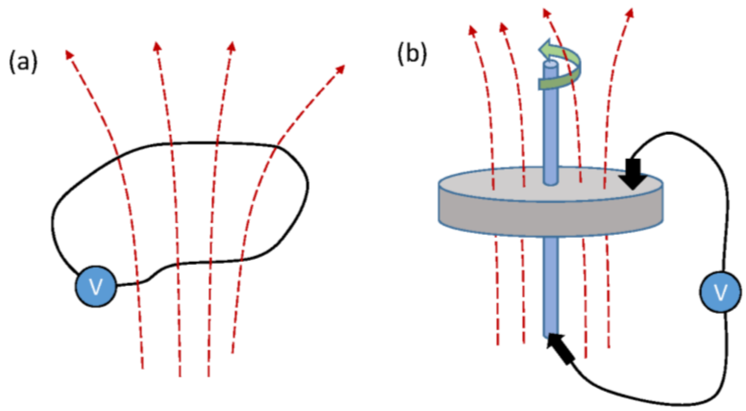


Figure 3. Derivation of the Faraday's law of electromagnetic induction for cases if the integral path is (a) a thin wire loop that does not intercept with any conductive medium; and (b) the integral path that intercepts with a conductive disc that is rotating around an axis. For both cases, the integral loop can be moving at a speed in space.

### 5.2 Extension of Faraday's law of electromagnetic induction in a moving large conductive media with considering inertia force

If the medium is moving with an acceleration, so that the inertia force may have to be considered once we consider the total force acting on the unit charge by standing on the rest reference frame of the circuit. In such a case, the force acting on a unit charge $q$ has to include the inertia force, thus Eq. (22) may have to be modified as:

$$q\boldsymbol{E}' - \frac{\partial}{\partial t}(m\boldsymbol{v}) = q\boldsymbol{E} + q\boldsymbol{v}_t \times \boldsymbol{B} \tag{27}$$

where $m$ is the total mass of the unit charge. The Faraday's law may be written as:

$$\oint_C \left[\boldsymbol{E} + \boldsymbol{v}_r \times \boldsymbol{B} + \frac{1}{q}\frac{\partial}{\partial t}(\mathrm{m}\boldsymbol{v})\right] \cdot d\boldsymbol{L} \approx - \iint_C \frac{\partial}{\partial t}\boldsymbol{B} \cdot d\boldsymbol{s} \tag{28a}$$

Using the Stokes' theorem, we have

$$\nabla \times (\boldsymbol{E} + \boldsymbol{v}_r \times \boldsymbol{B}) = - \frac{1}{q}\frac{\partial}{\partial t}[\nabla \times (\mathrm{m}\boldsymbol{v})] - \frac{\partial}{\partial t}\boldsymbol{B} \tag{28b}$$

We now discuss the following cases.

a) If the integral path is a loop circuit that does not intercept with a large conductive plate, so that the charge relative moving velocity is parallel to the integral path, the second term at the left-hand side of Eq. (28a) vanishes. The second condition, if the loop moving velocity $\boldsymbol{v}$ is a rigid translation, $\boldsymbol{v}(\mathrm{t})$, $[\nabla\times(\mathrm{m}\boldsymbol{v})] = 0$, the third term at the left-hand side of Eq. (36a) vanishes as well, we have

$$\nabla\times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{29}$$

b) If the integral path is a loop circuit that intercepts with a large conductive plate, so that the charge relative moving velocity may not be parallel to the integral path within the conductive disc, the second term at the left-hand side of Eq. (28a) remains. The second condition, if the loop moving velocity $\boldsymbol{v}$ is a rigid translation, $\boldsymbol{v}(\mathrm{t})$, $[\nabla\times(\mathrm{m}\boldsymbol{v})] = 0$, the third term at the left-hand side of Eq. (28a) vanishes as well, we have the following equation for the space inside the conductive medium:

$$\nabla\times (\boldsymbol{E} + \boldsymbol{v}_r\times\boldsymbol{B}) = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{30}$$

For the space outside of the media, Eq. (29) should be used.

c) For a general case in which the circuit loop has a time dependent and position dependent moving velocity, $\boldsymbol{v}(\mathbf{r},\mathrm{t})$, the term $\frac{\partial}{\partial t}[\nabla\times(\mathrm{m}\boldsymbol{v})]$ is related to the rotation movement of the loop circuit. The accelerated rotation motion of a shape-deformable circuit (e.g., a liquid or elastic loop, or expandable loop) can produce an electric field. Therefore, the term $\frac{\partial}{\partial t}[\nabla\times(\mathrm{m}\boldsymbol{v})]$ characterizes a "source" of electromagnetic radiation owing to the accelerated rotation of the "expandable" loop circuit. The solution for this case can follow the method introduced in Section 6.

## 6 General approach for solving the Maxwell's equations for a mechano-driven system

We now present a general solution of the Maxwell's equations for a mechano-driven moving medium system. If $\delta\boldsymbol{v}(t)$ is only a function of time for media that is rigid, so that the media movement is a solid translation, e.g. no rotation, assuming that the items of $\alpha\,\varepsilon\nabla\cdot(\boldsymbol{v}_0\times\boldsymbol{B})$ and $\alpha\,\varepsilon\nabla(\boldsymbol{v}_0\cdot\boldsymbol{E})$ are small in comparison to the source terms so that they can be neglected, Eqs. (20a-d) are approximated as

$$\varepsilon\nabla\cdot\boldsymbol{E} \approx \rho' \tag{31a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{31b}$$

$$\nabla\times (\boldsymbol{E} - \delta\boldsymbol{v}\times\boldsymbol{B}) = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{31c}$$

$$\nabla\times (\boldsymbol{B}/\mu + \varepsilon\delta\boldsymbol{v}\times \boldsymbol{E}) \approx \boldsymbol{J}' + \rho_f\,\delta\boldsymbol{v} + \varepsilon\frac{\partial}{\partial t}\boldsymbol{E} \tag{31d}$$

where:

$$\rho' = \rho_f - \nabla \cdot \boldsymbol{P}_{\mathrm{s}} \tag{32a}$$

$$\boldsymbol{J}' = \boldsymbol{J}_f + [\frac{\partial}{\partial t} + (\delta\boldsymbol{v} \cdot \nabla)]\boldsymbol{P}_{\mathrm{s}} \tag{32b}$$

By using:

$$\nabla\times (\delta\boldsymbol{v}\times \boldsymbol{E}) = \rho_f\,\delta\boldsymbol{v}/\varepsilon - (\delta\boldsymbol{v} \cdot \nabla)\boldsymbol{E} \tag{33a}$$

$$\nabla\times (\delta\boldsymbol{v}\times \boldsymbol{B}) = - (\delta\boldsymbol{v} \cdot \nabla)\boldsymbol{B} \tag{33b}$$

Eqs. (33a-d) become

$$\varepsilon\nabla \cdot \boldsymbol{E} = \rho' \tag{34a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{34b}$$

$$\nabla\times \boldsymbol{E} = -\frac{D}{Dt}\boldsymbol{B} \tag{34c}$$

$$\nabla\times \boldsymbol{B} = \mu\boldsymbol{J}' + \mu\varepsilon\frac{D}{Dt}\boldsymbol{E} \tag{34d}$$

where:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\delta\boldsymbol{v} \cdot \nabla) \tag{34e}$$

The law of conservation of charges is

$$\nabla\cdot\boldsymbol{J}' + \frac{D}{Dt}\rho_f = 0. \tag{35}$$

We now look into the solutions of Eqs. (34a-d). The $\boldsymbol{E}$ and $\boldsymbol{B}$ can be calculated by introducing the vector magnetic potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \nabla\times \boldsymbol{A} \tag{36a}$$

and the scalar electric potential $\varphi$ for electrostatics, we define

$$\boldsymbol{E} = -\nabla\varphi - \frac{D}{Dt}\boldsymbol{A} \tag{36b}$$

Substitute Eqs. (36a-b) into Eq. (34a-d), we have,

$$\nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{D^2}{Dt^2}\boldsymbol{A} = -\mu \boldsymbol{J}' \tag{37a}$$

$$\nabla^2 \varphi - \mu\varepsilon \frac{D^2}{Dt^2}\varphi = -\frac{\rho'}{\varepsilon} \tag{37b}$$

Under condition:

$$\nabla\cdot\boldsymbol{A} + \mu\varepsilon \frac{D}{Dt}\varphi = 0 \tag{37c}$$

where

$$\frac{D^2}{Dt^2} = [\frac{\partial}{\partial t} + (\delta\boldsymbol{v}\cdot\nabla)][\frac{\partial}{\partial t} + (\delta\boldsymbol{v}\cdot\nabla)] = \frac{\partial^2}{\partial t^2} + (\delta\boldsymbol{v}\cdot\nabla)\frac{\partial}{\partial t} + \frac{\partial}{\partial t}(\delta\boldsymbol{v}\cdot\nabla) + (\delta\boldsymbol{v}\cdot\nabla)(\delta\boldsymbol{v}\cdot\nabla)$$

$$\approx \frac{\partial^2}{\partial t^2} + (\delta\boldsymbol{v}\cdot\nabla)\frac{\partial}{\partial t} + \frac{\partial}{\partial t}(\delta\boldsymbol{v}\cdot\nabla) \tag{37d}$$

Eqs. (37a-d) are the generalized wave equations with the correction made by the non-inertia term $\delta\boldsymbol{v}$. The term $\frac{\partial}{\partial t}\delta\boldsymbol{v}$ is the acceleration, which represents the applied external force.

## 7 Maxwell's equations for a mechano-driven system in tensor format

We now transform Eqs. (34a-e) into the format of tensors, and the Maxwell's equations for a mechano-driven system is expressed in the standard format for field theory. We now use the standard expressions of following quantities for electrodynamics, the anti-symmetric strength tensor of electromagnetic field, except using the newly defined time operator in Eq. (34e),

$$F^{\alpha\beta} = \xi^{\alpha}A^{\beta} - \xi^{\beta}A^{\alpha} \tag{38a}$$

$$F_{\alpha\beta} = \xi_{\alpha}A_{\beta} - \xi_{\beta}A_{\alpha} \tag{38b}$$

where the newly defined operators are

$$\xi^{\alpha} = (\frac{1}{c}\frac{D}{Dt}, -\boldsymbol{\nabla}) \tag{39a}$$

$$\xi_{\alpha} = (\frac{1}{c}\frac{D}{Dt}, \boldsymbol{\nabla}) \tag{39b}$$

$$A^{\alpha} = (c\varphi, \boldsymbol{A}) \tag{39c}$$

$$A_{\alpha} = (c\varphi, -\boldsymbol{A}) \tag{39d}$$

One can prove

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{40a}$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{40b}$$

For an easy exercise, one can easily prove

$$E_x = -\frac{\partial}{\partial x}\varphi - \frac{D}{Dt}A_x = -(\xi^0 A^1 - \xi^1 A^0) \tag{41a}$$

$$B_x = \frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y = -(\xi^2 A^3 - \xi^3 A^2) \tag{41b}$$

Eqs. (34a-e) can be restated as:

$$\xi_\alpha F^{\alpha\beta} = \mu J^\beta \tag{42}$$

where $J^\beta = (c\rho', \boldsymbol{J}')$. This is the Maxwell's equations for a mechano-driven system. Note Eq. (48) is the same as that for the standard Maxwell's equations [15] except the operator $\partial_\alpha$ is replace by $\xi_\alpha$.

If $\delta\boldsymbol{v}$ = 0, Eq. (42) resumes the form of standard Maxwell's equations.

## 8 The Lagrangian of the Maxwell's equations for a mechano-driven system

We now derive the Lagrangian $L$ for the Maxwell's equations for a mechano-driven system. Λ is assumed to be a function of the density of the Lagrangian of the system $\Lambda(A_\alpha, \xi_\alpha A_\beta)$. We vary the action

$$\delta\int_{-\infty}^{\infty} L\, dt = \delta\iint_{-\infty}^{\infty} \Lambda(A_\alpha, \xi_\alpha A_\beta)\, d\boldsymbol{r}dt = 0 \tag{43}$$

which gives

$$\iint_{-\infty}^{\infty}\left[\frac{\partial\Lambda}{\partial A_\alpha}\delta A_\alpha + \frac{\partial\Lambda}{\partial(\xi_\alpha A_\beta)}\delta(\xi_\alpha A_\beta)\right] d\boldsymbol{r}dt = 0 \tag{44}$$

Now we look at the second term and integrate by part over (ct, x, y, z) [e.g. ($x_0$, $x_1$, $x_2$, $x_3$)], respectively, with considering the vanishing of the function at infinity. If the medium motion is a rigid translation $\delta\boldsymbol{v}(t)$ that is only time dependent, we have

$$\iint_{-\infty}^{\infty}[\frac{\partial\Lambda}{\partial A_\alpha}\delta A_\alpha]\, d\boldsymbol{r}dt - \iint_{-\infty}^{\infty}\{\,\xi_\alpha\left[\frac{\partial\Lambda}{\partial(\xi_\alpha A_\beta)}\right]\delta A_\beta\}\, d\boldsymbol{r}dt \tag{45}$$

We have the Lagrangian relation:

$$\frac{\partial\Lambda}{\partial A_\beta} - \xi_\alpha\frac{\partial\Lambda}{\partial(\xi_\alpha A_\beta)} = 0 \tag{46}$$

The density of the Lagrangian for the electromagnetic field is given by [15]

$$\Lambda = F^{\alpha\beta}F_{\alpha\beta} + \mu J^\alpha A_\alpha \tag{47}$$

Substituting Eq. (47) into Eq. (46), we have the Maxwell's equations for a mechano-driven system

$$\xi_\alpha F^{\alpha\beta} = \mu J^\beta \tag{48}$$

## 9 Discussions and Conclusions

There are two fundamental space and time: relativistic space and time, and absolute space and time. Therefore, there are two approaches for deal with the electrodynamics of moving media (Fig. 4). If the moving velocity is uniform in inertia frame, special relativity can be easily applied to this case without assuming low-moving speed limit. In this system, the total energy of electricity and magnetism is conservative. In special relativity, the general approach is starting from the integrated form of the four physics laws, a set of differential equations is derived (Maxwell's equations) for stationary media whose shape and boundary are independent of time. Then, the electromagnetic behavior of a moving media to be observed in Lab is described using the Lorentz transformation. Such coordination transformation is taken as the formal approach for moving media system. However, for a media system that moves in non-inertial frame, which means that the speed is a function of time at least, the theory for general relativity may be required for such a case, which is probably too complicated to be used for engineering purposes. For the speed we care about and the engineering purpose on earth, we believe that there is no need to use special relativity for applied electromagnetism.

In approach 1, if we start using the Galilean transformation instead of the Lorentz transformation, we could get the results of the Galilean electromagnetism [**Error! Bookmark not defined.**,7], in which the field in space at a time t is taken as a quasi-static case, e.g. the "frozen" field assumption. Therefore, the media distribution and related fields are treated "frame by frame" (as in films for a movie) under quasi-static approximation. The theory of moving media can be treated frame by frame with the use of constitutive relationships under slow-media moving cases. More approximations can be made for magnetic-dominated or electric-dominated systems. Again, such theory can be easily applied if the moving speed is a constant along a straight line, but we have to check if it can be applied for complex media moving trajectory cases, because we are not sure if the constitutive relationship derived using special relativity would hold in non-inertial frame.

The second approach is to directly starting from the four physics laws by deriving all of the fields in the Lab frame and in Lab coordination system using the Galilean space and time without making a coordination transformation [4,5,**Error! Bookmark not defined.**]. The most important advantage of this approach is that it can be applied to any media that move along complex trajectories in non-inertial frame as long as the moving speed is low and the relativistic effect is ignored. Such equations should not be Lorentz covariant simply due to the energy input from mechanical triggering and accelerated media motion. This approach is more effective for applied physics, which has been widely used in engineering electrodynamics [16, 17].

Lastly, we have introduced the extended form of the Faraday's law with considering the relative moving velocity of the charge with the presence of a large conductive medium and even with the presence of the inertia force produced by the movement of the reference frame. In all of the text books, the third equation in the Maxwell's equations is for a case that the conductive medium is a linear circuit. However, if the conductive medium is a plane or sheets in which the unit charge can drift off the integral path, an additional term has to be included in the Faraday's law for electromagnetic induction. More rigorously, if the inertia force is included and if it is significant, another term has to be included as well (Eq. 52). Our discussion does indicate that the Faraday's law of electromagnetic induction needs to be expanded for media that move with an acceleration, although the terms introduced may be small.

We need to point out that there are two formats of the integral equations for the Faraday's electromagnetic induction law: one is given in Eq. (6), and the other is given in Eq. (49):

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{L} = -\frac{d}{dt}\iint_C \boldsymbol{B} \cdot d\boldsymbol{s} \tag{49}$$

Equations (6) has been frequently used in text books that are for field theory, and it is built based on the electromotive force. Eq. (49) is more often used for engineering electrodynamics [16, 17]. We believe that the results received from both statements would be about the same if one keeps consistent within its own theoretical frame and as long as we are confined to the engineering applications on earth, provided the theory is self-consistent.

In conclusion, we have elaborated that, for the electromagnetic phenomena we are care about for macroscopic objects on earth, the Galilean transformation is accurate enough. An approximated method is introduced to expand the Maxwell's equations for a mechano-driven system into non-inertia frame, which may resume the format of the standard Maxwell's equations if the media moving velocity is a constant. We found that the standard form of Maxwell equations holds in inertia frame, but in non-inertia frame, the Faraday's law of electromagnetic induction and the Ampere-Maxwell law may not be stated as the differential forms that we are familiar with. Therefore, one has to be careful if using them for dealing with media movement with acceleration. An approximate and effective approach needs to be developed. This is just the objective of deriving the Maxwell's equations for a mechano-driven slow-moving media system. Lastly, the theory is presented in Lagragian scheme, from which the Maxwell's equations for a mechano-driven media is derived.

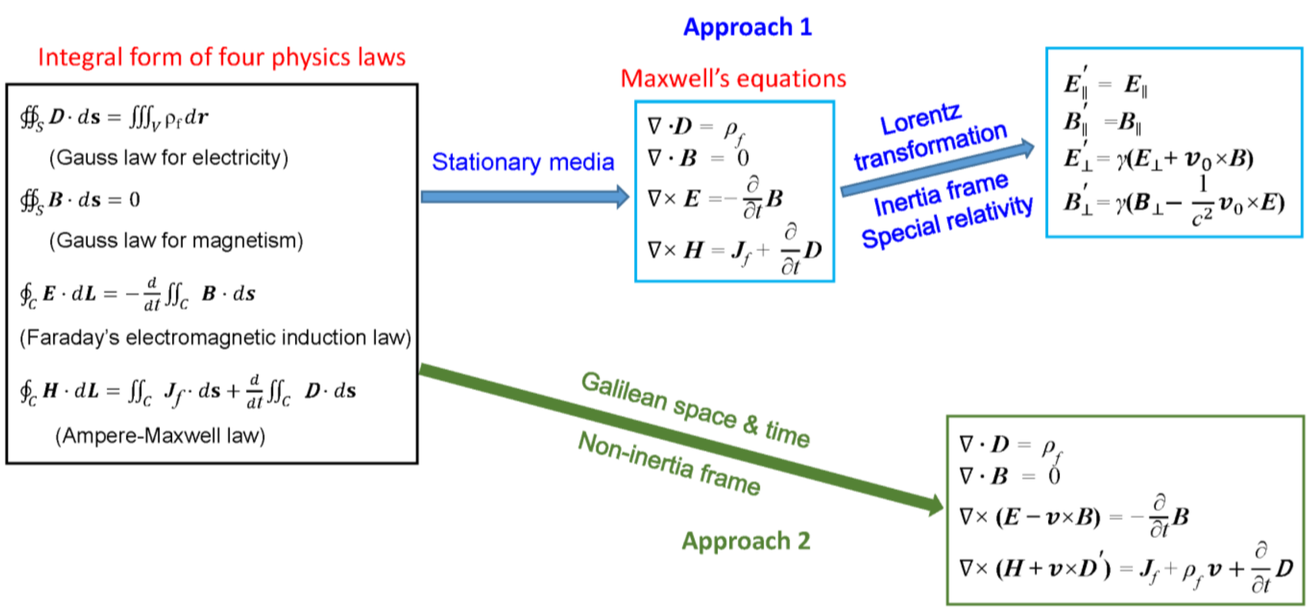

Figure 4. Two fundamental approaches for developing the electrodynamics of a moving media system. The first approach is special relativity through the Lorentz transformation. The second approach is directly derivation from the integral form of the four physics laws in Galilean space and time.

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